# Integration of forms over oriented manifolds (6.4)

## Defn's:

### ① Orientation preserving linear transformation

Let $V \subset \mathbb{R}^n$ be a $k$-dimensional subspace with orientation $\Omega : \mathcal{B}(V) \to \{+1, -1\}$.

A linear transformation $T : \mathbb{R}^k \to V$ preserves orientation if $\Omega(T(\vec{e}_1), \dots, T(\vec{e}_n)) = +1$

Otherwise it is orientation reversing.

### ② Orientation-preserving parametrization of a manifold

Let $M$ be a $k$-dimensional manifold oriented by $\Omega$ & let $U \subset \mathbb{R}^k$ be a subset with boundary of $k$-dim volume 0. Let $\gamma : U \to M$ parametrize $M$ (and let $X$ be as before in the def'n of parametrization)

Then $\gamma$ is orientation preserving if for all $u \in U - X$:

linear trans. $\to [D\gamma(u)]\vec{e}_1$

$$\Omega(\vec{D}_1\gamma(u), \dots, \vec{D}_k\gamma(u)) = +1$$

form a basis for $T_{\gamma(u)}M$

## Example

$\gamma_1(\theta) = (\cos\theta, \sin\theta)$ &

$\gamma_2(\theta) = (\sin\theta, \cos\theta)$

$\theta \in [0, 2\pi]$ are parametrizations of the unit circle.

Consider the orientation induced by $\vec{E}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$

i.e. $\Omega^{\vec{E}}(\vec{v}) = \text{sign}\left(\vec{E}(\vec{x}) \cdot \vec{v}\right)$

Then $\Omega^{\vec{E}}(\gamma_1'(\theta)) = \text{sign}\left((-\sin\theta, \cos\theta) \cdot (-\sin\theta, \cos\theta)\right)$

$= 1 \Rightarrow \gamma_1$ is orien preserving,

& $\Omega^{\vec{E}}(\gamma_2'(\theta)) = \text{sign}\left((-\cos\theta, \sin\theta) \cdot (\cos\theta, \sin\theta)\right)$,

$= -1 \Rightarrow \gamma_2$ is orientation reversing.

## Integrating forms over oriented manifolds

Let $M \subset \mathbb{R}^n$ be a $k$-dimensional oriented manifold & let $\gamma_1 : U_1 \to \mathbb{R}^n$

$\gamma_2 : U_2 \to \mathbb{R}^n$

be two orientation preserving parametrizations of $M$. $\forall k$-form $\varphi$:

$$\int_{[\gamma_1(U_1)]} \varphi = \int_{[\gamma_2(U_2)]} \varphi .$$

**Remark:** If $\gamma_1, \gamma_2$ are both orientation reversing same result.

If one is preserving & the other reversing then $\int_{[\gamma_1(U_1)]} \varphi = -\int_{[\gamma_2(U_2)]} \varphi$

**Definition**

$k$-form

$$\int_M \varphi := \int_{[\gamma(U)]} \varphi = \int_U \varphi\left(P_{\gamma(u)}\left(\vec{D}_1\gamma(u), \dots \vec{D}_k\gamma(u)\right)\right) |d^k u|$$

$M$: manifold

$[\gamma(U)]$: orientation preserving parametrization

**Example:** Let $\varphi = y\,dy\wedge dz + x\,dx\wedge dz$

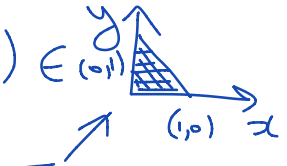
Let $P$ be the part of the plane $x+y+z=1$ with $x\geq 0, y\geq 0, z\geq 0$ oriented by the normal $\vec{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Compute $\int_P \varphi$

Sol'n

① Let's parametrize $P$ by $\gamma: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \\ 1-x-y \end{pmatrix}$

where $(x,y) \in T$




② Let's check if it's orientation preserving:

$$\det\left([\vec{n}, D_1\gamma, D_2\gamma]\right) = \det\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}\right) > 0$$

③
$$\int_P \varphi = \int_{[\gamma(u)]} \varphi$$

$$= \int_T \left(y\,dy\wedge dz + x\,dx\wedge dz\right)\left(P_{\begin{bmatrix} x \\ y \\ 1-x-y \end{bmatrix}}\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)\right) |dx\,dy|$$

$$= \int_0^1 \int_0^{1-y} y \det\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} + x \det\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} dx\,dy$$

$$= \int_0^1 \left(\int_0^{1-y} y - x\,dx\right) dy = \cdots$$

# 6.5) Forms & vector calculus

We are now ready to start connecting forms with some more "physically intuitive" concepts that we may be more familiar with (particularly in $\mathbb{R}^3$).

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**Def'n**

A line integral of a vector field

$$\vec{F} = (P(x,y,z), Q(x,y,z), R(x,y,z))$$

in $\mathbb{R}^3$ along an oriented curve $\vec{c}: [a,b] \to \mathbb{R}^3$ is defined as

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} := \int_{\vec{c}} P\,dx + Q\,dy + R\,dz$$

1-form called "the Work form"

**Remarks:** ① $W_{\vec{F}} = P\,dx + Q\,dy + R\,dz$ is called the work form of the vector field $\vec{F}$.

(2) As a 1-form in $\mathbb{R}^3$, $W_{\vec{F}}$ takes a vector, say $\vec{V} = (v_1, v_2, v_3) \in \mathbb{R}^3$ and returns

$$P(x,y,z)v_1 + Q(x,y,z)v_2 + R(x,y,z)v_3 = \vec{F} \cdot \vec{V}$$

So we may write $\boxed{W_{\vec{F}}(\vec{v}) = W_{\vec{F}}\left(P_{\vec{x}}(\vec{v})\right) = \vec{F}(\vec{x}) \cdot \vec{v}}$ (*)

(3) This means that we can write

$$\int_{\vec{C}} \vec{F} \cdot d\vec{s} = \int_{\vec{C}} \vec{W}_F$$ → this is just $P\,dx + Q\,dy + R\,dz$ after all

$$= \int_{[a,b]} W_{\vec{F}}\left(P_{\gamma(u)}(\vec{D}_1 \gamma(u))\right) |du|$$

$$= \int_a^b (P\,dx + Q\,dy + R\,dz)\left(P_{\gamma(u)}(\gamma'(u))\right) du$$

$$\boxed{\int_{\vec{C}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\gamma(u)) \cdot \vec{\gamma}'(u)\, du} \quad \text{by } (*)$$

## So, why is $W_{\vec{F}}$ called the work form?

By definition,

$$\int_{\vec{C}} W_{\vec{F}} = \int_{\vec{C}} \vec{F} \cdot d\vec{s}$$

$$= \int_a^b \vec{F}(\gamma(t)) \cdot \vec{\gamma}'(t)\, dt$$

is the amount of work done by a "force" $\vec{F}$ along an oriented curve $\vec{C}$.

**Example:** Let $\vec{c}(t) = (\cos t, \sin t, t)$, $0 \le t \le 2\pi$ be a curve oriented by $\vec{c}'(t)$. Let $\vec{F}(x,y,z) = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$.

Compute $\int_{\vec{C}} W_{\vec{F}}$

**Soln:** $\int_{\vec{C}} W_{\vec{F}}$

$$= \int_0^{2\pi} (x^2 dx + y^2 dy + z^2 dz)\left(P_{(\cos t, \sin t, t)}(-\sin t, \cos t, 1)\right) dt$$

$$= \int_0^{2\pi} \cos^2 t \cdot (-\sin t) + \sin^2 t \cdot \cos t + t \, dt = 0 + 0 + \frac{4\pi^2}{2}$$

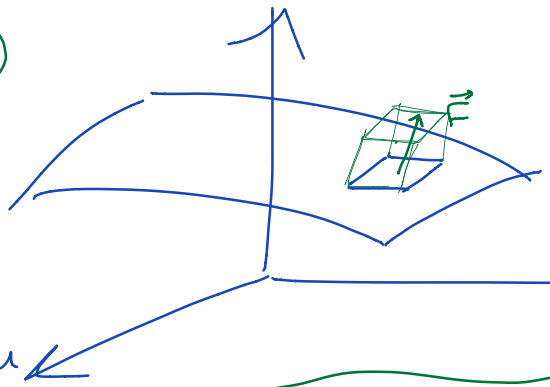
**Def'n** (Surface Integrals of vector fields)

The Surface Integral of $\vec{F}$ over an oriented surface $S$ parametrized by $\gamma: T \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\vec{u} \mapsto S$ ($\vec{u} = (u, v)$) is defined as

$$\iint_S \vec{F} \cdot d\vec{S} := \iint_T \det\left(\left[\vec{F}, \vec{D}\gamma|_{\vec{u}}, \vec{D}_2\gamma|_{\vec{u}}\right]\right) du\, dv$$

What does this mean?

Ex: If $\vec{F}$ is the vector field describing the velocity of water in $\mathbb{R}^3$, and if $S = (u, v, f(u,v))$ is a surface, that the water is flowing through, then $\iint_S \vec{F} \cdot d\vec{S}$ is the volume of water flowing through $S$ per unit of time



## Remarks:

① Defining the Flux Form

$$\Phi_{\vec{F}}(\vec{v},\vec{w}) = \Phi_{\vec{F}}\left(P_{\vec{x}}(\vec{v},\vec{w})\right)$$
$$= \det\left[\vec{F}(\vec{x}),\vec{v},\vec{w}\right]$$

we see that

$$\iint_S \vec{F}\cdot d\vec{S} = \int_S \Phi_{\vec{F}}$$
$$= \int_U \Phi_{\vec{F}}\left(P_{\vec{\gamma}(\vec{u})}\left(\vec{D}_1\gamma(u), \vec{D}_2\gamma(u)\right)\right)|d^2u|$$
$$= \int_U \det\left[\vec{F}(\underbrace{\gamma(u)}_{\vec{x}}), \vec{D}_1\gamma|_{\vec{u}}, \vec{D}_2\gamma|_{\vec{u}}\right] |d^2u|$$

② You can check that if $\vec{F}(x,y,z) = \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix}$

$\Phi_F$ is a 2-form field

Then $\Phi_{\vec{F}} = F_1\, dy\wedge dz + F_2\, dz\wedge dx + F_3\, dx\wedge dy$

Example: Let $\vec{F}(x,y,z) = (x,y,z)$
and let $S$ be the sphere of radius 1
oriented by the outward normal

Compute the flux of $F$ through $S$, i.e.,
compute $\iint_S \vec{F} \cdot d\vec{S}$ (equivalently $\int_S \varphi_{\vec{F}}$)

Solution: ① Let's parametrize $S$ by

$$\gamma : (\theta, \varphi) \to (\cos\theta \sin\varphi,\ \sin\theta \sin\varphi,\ \cos\varphi)$$

where $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$

$$\Rightarrow D_1\gamma = (-\sin\theta \sin\varphi,\ \cos\theta \sin\varphi,\ 0)$$
$$D_2\gamma = (\cos\theta \cos\varphi,\ \sin\theta \cos\varphi,\ -\sin\varphi)$$

② Check the orientation:
the outward normal is $\vec{n} = (x,y,z)$
$$= (\cos\theta \sin\varphi,\ \sin\theta \sin\varphi,\ \cos\varphi)$$

$$\Rightarrow \Omega^{\vec{n}}(D_1\gamma, D_2\gamma) = \operatorname{sign}(\det([\vec{n}, \vec{D}_1\gamma, \vec{D}_2\gamma])) < 0$$

because

$$\det\begin{bmatrix} \cos\theta\sin\varphi & -\sin\theta\sin\varphi & \cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \sin\theta\cos\varphi \\ \cos\varphi & 0 & -\sin\varphi \end{bmatrix}$$

$\vec{n}$ | $D_1\gamma$ | $D_2\gamma$

$$= -\cos^2\theta\sin^3\varphi + \sin\theta\sin\varphi\left(-\sin\theta\sin^2\varphi - \sin\theta\cos^2\varphi\right)$$
$$+ \cos\theta\cos\varphi\left(-\cos\theta\sin\varphi\cos\varphi\right)$$

$$= -\cos^2\theta\sin^3\varphi - \sin^2\theta\sin^3\varphi - \sin^2\theta\sin\varphi\cos^2\varphi$$
$$- \cos^2\theta\sin\varphi\cos^2\varphi$$

$$= -\sin^3\varphi - \sin\varphi\cos^2\varphi = -\sin\varphi < 0 \quad \text{when}$$
$$\varphi \in (0, \pi)$$

③ Since our parametrization was orientation reversing, we compute

$$\int_S \varphi_F = -\iint \det\left[\vec{F}, D_1\gamma, D_2\gamma\right] d\theta\, d\varphi$$

Fortunately, $\vec{F} = \vec{n}$, so we know from ② for this problem

that $\int_S \varphi_F = -\int_0^{2\pi}\int_0^{\pi} -\sin\varphi \, d\varphi \, d\theta = +4\pi$ ///

## So far we've seen

- $W_{\vec{F}} = F_1\,dx + F_2\,dy + F_3\,dz \rightsquigarrow$ 1-form field (work form)

  $\left(\vec{F}(x,y,z) = \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix}\right)$

- $\phi_{\vec{F}} = F_1\,dy\wedge dz + F_2\,dz\wedge dx + F_3\,dx\wedge dy$

  $\hookrightarrow$ 2-form field (flux form)

- Next, we'll see 3-form field (mass form)

**Defn:** Let $f: U \subset \mathbb{R}^3 \to \mathbb{R}$ be a function

The mass form $M_f(P_x(\vec{v}_1,\vec{v}_2,\vec{v}_3))$

$:= f(\vec{x}) \det[\vec{v}_1,\vec{v}_2,\vec{v}_3]$

Now: Let $\gamma: U \subset \mathbb{R}^3 \to V \subset \mathbb{R}^3$ be a $C^1$ map preserving the standard orientation

$$\int_V M_f = \int_U M_f\left(P_{\gamma(u)}\left(\vec{D}_1\gamma|_u, \vec{D}_2\gamma|_u, \vec{D}_3\gamma|_u\right)\right) |d^3u|$$

"mass of $V$"

$$= \int_U f(\vec{x}) \underbrace{\det\left[\vec{D}_1\gamma(u), D_2\gamma(u), D_3\gamma(u)\right]}_{\text{volume of parm}} |d^3u|$$

$f(\vec{x})$: Can think of this as density (mass density, charge density)

Example:

Let $B$ be the unit ball in $\mathbb{R}^3$ ($B = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$) and let

Let $f(x,y,z) = 2 - \sqrt{x^2+y^2+z^2}$ be its charge density. Find the charge of $B$.

Sol'n:

① Parametrize $B$:

$\gamma: (r, \theta, \varphi) \longrightarrow (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi)$

where $0 \le r \le 1$, $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$

② Check whether $\gamma$ is orientation preserving (answer: no, similar to last example)

③ $\int_V M_f$

$$= -\int_U 2-\sqrt{x^2+y^2+z^2}\ \det \begin{bmatrix} \cos\theta\sin\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\cos\theta\cos\varphi & r\sin\theta\cos\varphi \\ \cos\varphi & 0 & -r\sin\varphi \end{bmatrix} dr\,d\theta\,d\varphi$$

$A$

Since $\det A$

$= r^2\cos^2\theta\sin^3\varphi + r^2\sin\theta\sin\varphi(-\sin\theta\sin^2\varphi - \sin\theta\cos^2\varphi)$
$+ r^2\cos\theta\cos\varphi(-\cos\theta\sin\varphi\cos\varphi)$

(Same as previous example $\times r^2$)

$$= -\left(\cos^2\theta \sin^3\varphi - \sin^2\theta \sin^3\varphi - \sin^2\theta \sin\varphi \cos^2\varphi - \cos^2\theta \sin\varphi \cos^2\varphi\right) r^2$$

$$= \left(-\sin^3\varphi - \sin\varphi\cos^2\varphi\right) r^2 = -\sin\varphi\, r^2$$

$$\int_V M_F = +\iiint_{\text{Ball}} (2-r) r^2 \sin\varphi \, dr\, d\theta\, d\varphi$$

$$= 2\pi \int_0^{\pi}\int_0^{1} 2r^2\sin\varphi - r^3\sin\varphi \, dr\, d\varphi$$

$$= 2\pi \cdot \left[\frac{-2}{3}\cos\varphi\Big|_0^{\pi} - \frac{1}{4}(-\cos\varphi)\Big|_0^{\pi}\right]$$

$$= \frac{4}{3}\pi - \pi = \pi/3$$

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## Work, flux, and mass forms in $\mathbb{R}^n$

- 0-form fields are functions
- Every 1-form field is the work form of a vector field
- Every $n-1$ form field is the flux form of a vector field
- Every $n$-form field is the mass form of a fluid.

**Defn:** If $\vec{F}$ is a vector field on $\mathbb{R}^n$ Then the flux form $\Phi_{\vec{F}}$ is given by

$$\Phi_{\vec{F}} P_{\vec{x}}(\vec{v}_1, \ldots, \vec{v}_{n-1}) = \det[\vec{F}(x), \vec{v}_1, __, \vec{v}_{n-1}]$$

**Defn:** Let $f: \mathbb{R}^n \to \mathbb{R}$ be a fct

The mass form $M_f := f\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$

**Proposition:** on $\mathbb{R}^3$

$$\Phi_{\vec{F} \times \vec{G}} = W_{\vec{F}} \wedge W_{\vec{G}}$$

$$M_{\vec{F} \cdot \vec{G}} = W_{\vec{F}} \wedge \Phi_{\vec{G}}$$

$$= W_{\vec{G}} \wedge \Phi_{\vec{F}}$$

**Proof:** part of HW/exercise

## Example (Fun with Physics)

The force experienced by a charge $q$ at $(\vec{x}, t)$ is

$$\vec{F}(\vec{x},t) = q\vec{E}(\vec{x},t) + \left(\frac{\vec{v}}{c} \times \vec{B}(\vec{x},t)\right)$$

- $\vec{F}(\vec{x},t)$: Force
- $q$: Charge
- $\vec{E}(\vec{x},t)$: time-varying electric field
- $\frac{\vec{v}}{c}$: velocity of $q$ (divided by speed of light)
- $\vec{B}(\vec{x},t)$: time varying magnetic field

- **Remark:** $\vec{E}$ & $\vec{B}$ are *not* vector fields in $\mathbb{R}^4$.

To see this: suppose $\vec{E} = 0$ in one coordinate system, and so is $\vec{V}$.

Then $\vec{F} = 0$

- now: observe the system from a frame moving with velocity $\vec{V}$, $\vec{F}$ is still 0 (no acceleration is observed)

$\Rightarrow \vec{E} \neq 0$ so $\vec{E}$ is not a vector field!

(Because $\vec{D}$ should be conserved by a change of coordinates)

The Faraday: a 2-form field on $\mathbb{R}^4$ Given by

$$W_{\vec{E}} \wedge c\,dt + \Phi_{\vec{B}}$$

$$= (E_x\,dx + E_y\,dy + E_z\,dz) \wedge c\,dt$$
$$+ B_x\,dy \wedge dz + B_y\,dz \wedge dx$$
$$+ B_z\,dx \wedge dy .$$

(More on this later)